PHYSICAL REVIEW E VOLUME 49, NUMBER 3 MARCH 1994

Turbulence with an infinite number of conservation laws

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(Received 20 July 1993)

It is shown that if the pair correlation function of any tracer in incompressible turbulent flow is scale invariant with the exponent ζ_2, then the exponent of two-point function of $2n$th order does not equal $n\zeta_2$. In this case, the probability distribution should depend, generally speaking, on an infinite number of parameters (fluxes of the integrals). Three examples are considered: two-dimensional vorticity cascade, action cascade in Clebsch variables, and entropy cascade in inhomogeneously heated fluid.

PACS number(s): 47.10.+g, 47.27.Gs

I. INTRODUCTION

All one-time correlation functions of steady turbulence are presumed to be scale invariant in the inertial interval of scales. If the exponent ζ_n of the nth correlation function is not proportional to n, one calls this an anomalous scaling and usually refers to the intermittency phenomenon. This is a great challenge for theory to explain such phenomena directly from the equation of motion. This paper demonstrates one possible mechanism of an anomalous scaling related to the presence of an infinite number of integrals of motion.

Knowledge of the whole set of integrals is necessary to describe the statistics of a system both in equilibrium and in a strongly nonequilibrium (turbulent) state. In equilibrium, one defines Gibb's equipartition on the hypersurface of all constant integrals in phase space. In turbulence, a steady state is usually found by requiring the flux of the respective integral to be constant in k space provided the locality of interaction can be demonstrated. If there is only one integral, it is natural to assume that, for instance, all two-point correlation functions are determined by the constant flux of this integral and the distance between the points. This immediately gives a linear (normal) scaling: $\zeta_n \propto n$. The presence of an infinite number of integrals may give an infinite number of constraints. The turbulent probability distribution may thus depend on an infinite number of fluxes and be multiscaling. As shown in this paper, even an infinite number of conservation laws does not necessarily give intermittency: Anomalous scaling appears only in the cases when there are inertial intervals for all the integrals. The pumping of a high-order integral depends on the behavior of lower-order correlation functions and may be distributed over the whole k space even for spectrally narrow external force.

There may be other reasons for an anomalous scaling (such as nonlocality of the cascade that leads to the appearance of an external scaling in the expressions for the correlation functions) which are not considered in this paper.

II. GENERAL CONSIDERATION

This paper is devoted to the particular cases of hydrodynamic turbulence with an infinite number of integrals. Namely, we consider the advection equation for some scalar quantity $\theta(\mathbf{r},t)$,

$$\frac{\partial\theta(\mathbf{r},t)}{\partial t} + (\mathbf{v}\cdot\nabla)\theta(\mathbf{r},t) = p(\mathbf{r},t) - d(\mathbf{r},t)\ . \qquad (1)$$

If $\mathbf{v}$ is a velocity of an incompressible fluid so that $\operatorname{div}\mathbf{v}=0$, then Eq. (1) conserves $\int\theta^n d\mathbf{r}$ for any n without external pumping p and damping d. We are interested in nonlinear problems so that θ is somehow related to the velocity. In the examples below, θ will be the vorticity of two-dimensinoal (2D) velocity field, Clebsch variable, or temperature acting back on the velocity by a buoyancy force. A common feature of all three examples is that the integral over the whole space of any power of θ is an integral of motion of the unforced undamped problem.

If one considers forced turbulence, the way of pumping θ^2 differs from that of pumping high-order ($n>2$) integrals. Let us consider the external force p with a typical correlation scale L_p and the damping d acting on the scale L_d. Both p and d can be functionals of θ. We start from the direct cascade assuming $L_p \gg L_d$. For the flux of θ^2 stuff in the inertial interval of scales $L_d \ll |\mathbf{r}_1-\mathbf{r}_2| = r_{12} \ll L_p$, one obtains in a Kolmogorov manner

$$\langle[(\mathbf{v}_1\cdot\nabla_1)+(\mathbf{v}_2\cdot\nabla_2)]\theta_1\theta_2\rangle = \langle p_1\theta_2+p_2\theta_1\rangle = P_2\ . \quad (2)$$

The right-hand side is constant at $r_{12} \ll L_p$ and it is equal to the pumping rate of θ^2. The constancy of the flux fixes the scaling exponent of the correlation function on the left-hand side of (2). For θ^4 one gets similarly

$$\langle[(\mathbf{v}_1\cdot\nabla_1)+(\mathbf{v}_2\cdot\nabla_2)]\theta_1^2\theta_2^2\rangle = \langle p_1\theta_1\theta_2^2+p_2\theta_2\theta_1^2\rangle\ . \quad (3)$$

As noted by Lebedev [1], besides the irreducible part that is constant in the inertial interval, the correlator on the right-hand side of (3) necessarily contains the reducible parts $\langle p_1\theta_2\rangle\langle\theta_1\theta_2\rangle = P_2\langle\theta_1\theta_2\rangle$ that change with r_{12} as the pair correlator. If the absolute value of the r-dependent part of $\langle\theta_1\theta_2\rangle$ is a growing function of r_{12}, then the reducible (smeared in k space) part of the pumping decreases when the distance r_{12} decreases as one passes deep into the inertial interval. Therefore, the flux of θ^4 can be considered as a constant at $L_p(P_4/P_2^2)^{1/\zeta_2} \gg r_{12} \gg L_d$. Here P_4 is the pumping rate of θ^4. Inductively, one can employ such a consid-

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eration for the correlation function of any order. The constancy of the fluxes prescribes scaling exponents that are, generally speaking, different from those given by a linear scaling (this depends on the scaling of the vertex $v\nabla$). High-order conservation laws can thus be the reason for an anomalous scaling if the correlation function drop while passing to the inertial interval (as it will be shown below for turbulent convection and the Clebsch cascade).

Note that the pair correlation function approaches constant as $r_{12} \to 0$ in all known examples that possess scale invariance. If otherwise the correlation functions increase as we pass into the inertial interval (which corresponds to a logarithmic behavior, for instance), then the reducible parts of force correlators prevail so that the fluxes are not constant for $n > 2$. As a result, the higher-order correlators are determined by lower-order ones and no additional constraints appear. Let us emphasize that even if all the correlation functions of p are zero in the inertial interval in k space, the pumping of high-order integrals might be nonzero there. In particular, this is the case for 2D vorticity cascade which we briefly consider here.

III. VORTICITY CASCADE

The Euler equation of motion for incompressible two-dimensional fluid could be written for the vorticity $\omega(\mathbf{r},t) = \mathrm{curl}\,\mathbf{v}$

$$\frac{\partial \omega(\mathbf{r},t)}{\partial t} + (\mathbf{v}\cdot\nabla)\omega(\mathbf{r},t) = p(\mathbf{r},t) - d(\mathbf{r},t) \; . \tag{4}$$

The steady vorticity cascade corresponds to the pair correlations function [2,3]: $\langle \omega(\mathbf{r}_1)\omega(\mathbf{r}_2)\rangle \propto \ln^{2/3}(L/r_{12})$. Considering the flux of ω^4 according to (3), one gets

$$\langle [(\mathbf{v}_1\cdot\nabla_1) + (\mathbf{v}_2\cdot\nabla_2)]\omega_1^2\omega_2^2\rangle \propto \ln^{2/3}(L/r_{12}) \; . \tag{5}$$

By the same means one can show that the fluxes of any power of the vorticity (except the second one) are nonconstant in the inertial interval so that no additional constraints appear. The same is true for the passive scalar in a large-scale velocity field (Batchelor regime) where the correlation functions are also logarithmic.

It is worth emphasizing that fluxes change since there are no inertial intervals for higher integrals. This change has nothing to do with nonconservation, which is sometimes erroneously believed to be connected with a truncation of the Fourier representation. The same consideration of the high-order fluxes in the inertial interval can be provided in k space. To do this, one needs to establish the detailed conservation of the integrals. The conservation of the second-order integral (for any interacting triad of wavevectors) can be readily established due to the simple symmetry of the vertex. Here we show that this symmetry can be generalized for any set of interacting modes (quartet, quintet, etc.) so that any integral possesses detailed conservation (this problem was posed by Kraichnan [4]). This is relevant to numerical simulations of the 2D Euler equation by spectral methods.

The Euler equation in $\mathbf{k}$ representation

$$\frac{\partial \omega_k}{\partial t} = \int \Gamma_{12}\omega_1^*\omega_2^*\delta(\mathbf{k}+\mathbf{k}_1+\mathbf{k}_2)\, d^2k_1 d^2k_1 \tag{6}$$

has the vertex $\Gamma_{12} = \Gamma(\mathbf{k}_1,\mathbf{k}_2) = [\mathbf{k}_1\mathbf{k}_2](k_1^{-2} - k_2^{-2})/2$, which possesses an infinite number of symmetries. To show how they provide for the conservation of the integrals of motion, let us introduce the spectral densities of the integrals $H_n = \int \omega^n(\mathbf{r},t)d\mathbf{x}$

$$h_n(\mathbf{k}) = \omega_k \int \omega_1 \cdots \omega_{n-1}\delta(\mathbf{k}+\cdots+\mathbf{k}_{n-1})d^2k_1\cdots d^2k_{n-1} \; .$$

The spatial density (per unit volume) of the integral H_n is thus equal to $\int h_n(\mathbf{k})\, d^2k$. The conservation laws follow from the expression for the time derivative

$$\begin{aligned} \dot{h}_n(\mathbf{k}_1) &= \sum_{p=1}^{n} \int \omega(\mathbf{k}_1)\cdots\dot{\omega}(\mathbf{k}_p)\cdots\omega(\mathbf{k}_n)\, d^2k_2\cdots d^2k_n \\ &= \sum_{p=1}^{n} \int \omega(\mathbf{k}_1,t)\cdots\Gamma_{qp}\omega_q^*\omega_p^*\delta(\mathbf{k}_p+\mathbf{q}+\mathbf{p})\cdots \\ &\quad \times\omega(\mathbf{k}_n,t)\, d^2k_2\cdots d^2k_n d^2q d^2p \; . \end{aligned}$$

For example, the conservation of the squared vorticity

$$\begin{aligned} &\frac{\partial}{\partial t}\int h_2(\mathbf{k},t)\, d^2k \\ &\quad = 2\mathrm{Re}\int \Gamma_{12}\omega_k^*\omega_1^*\omega_2^*\delta(\mathbf{k}+\mathbf{k}_1+\mathbf{k}_2)\, d^2k d^2k_1 d^2k_2 \\ &\quad = 0 \end{aligned}$$

follows from the Jakobi identity for the vertex $(\Gamma_{pq} + \Gamma_{kp} + \Gamma_{kq})\delta(\mathbf{k}+\mathbf{q}+\mathbf{p}) = 0$. By the same means one gets

$$\begin{aligned} &\frac{\partial}{\partial t}\int h_3(\mathbf{k},t)\, d^2k \\ &\quad = \int \Gamma_{12}^*\omega_1^*\omega_2^*\omega_3^*\omega_4^*\delta(\mathbf{k}_1+\mathbf{k}_2+\mathbf{k}_3+\mathbf{k}_4)d^2k_1\ldots d^2k_4 \\ &\quad = 0 \end{aligned}$$

by virtue of the identity

$$(\Gamma_{12}+\Gamma_{13}+\Gamma_{14}+\Gamma_{23}+\Gamma_{24}+\Gamma_{34})\delta(\mathbf{k}_1+\mathbf{k}_2+\mathbf{k}_3+\mathbf{k}_4) = 0 \; .$$

Similar symmetry provides for the conservation of the nth vorticity integral:

$$\sum^{n+1}\Gamma_{ij}\delta\left(\sum_{p=1}^{n+1}\mathbf{k}_p\right) = 0 \; . \tag{7}$$

Here the first sum is taken over the different choices of noncoincident i and j. Note that if we restrict the dynamical vertex Γ into some subspace of k space then only the symmetry with $n = 2$ survives.

If the spectrum of 2D vorticity cascade was of power type (nonlogarithmic), then the presence of all the integrals imposes some restrictions that enable one to rule out hypothetical conformal solutions [5].

IV. BOLGIANO-OBUKHOV CASCADE IN TURBULENT CONVECTION

Now θ from Eq. (1) describes the temperature fluctuations acting back on the velocity field by the buoyancy force:

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v}\cdot\nabla)\mathbf{v} = -\beta\mathbf{g}\theta \; , \tag{8}$$

where β is the volume thermal expansion coefficient and $\mathbf{g}$ is the acceleration due to gravity. Equation (8) corresponds to the so-called Boussinesq approximation which is valid for not too strong temperature fluctuations. An unforced undamped system [Eqs. (1) and (8)] conserves the mechanical energy and any power of the temperature (beyond this approximation, any power of entropy, not of temperature, is conserved). Depending on the conditions of excitation, there may exist the interval of scales where turbulence is completely determined by the temperature flux [6] (physically this corresponds to the entropy flux from the viscous scale to the pumping region). In this case, a steady solution should have constant flux (2) while the flux of the kinetic energy depends on the scale due to conversion of the kinetic energy into the potential one:

$$\langle[(\mathbf{v}_1\cdot\boldsymbol{\nabla}_1)+(\mathbf{v}_2\cdot\boldsymbol{\nabla}_2)]\theta_1\theta_2\rangle = P_2 \ ,$$

$$\langle[(\mathbf{v}_1\cdot\boldsymbol{\nabla}_1)+(\mathbf{v}_2\cdot\boldsymbol{\nabla}_2)](\mathbf{v}_1\mathbf{v}_2)\rangle = \beta\langle\theta_1(\mathbf{g}\cdot\mathbf{v}_2)+\theta_2\cdot(\mathbf{g}\mathbf{v}_1)\rangle \ .$$

Assuming that these correlation functions are determined by two scaling exponents ζ and η of the temperature and velocity fields $\theta_1-\theta_2\propto r_{12}^{\zeta}$ and $|\mathbf{v}_1-\mathbf{v}_2|\propto r_{12}^{\eta}$, one gets Bolgiano-Obukhov scaling $\zeta=1/5, \eta=3/5$ [6–10]. In particular, it gives the pair correlation function

$$\langle(\theta_1-\theta_2)^2\rangle \propto P_2^{4/5} r_{12}^{2/5} \ , \tag{9}$$

which corresponds to the power spectrum of the temperature fluctuations $P(\omega)\propto\omega^{-7/5}$, which satisfactory fits the data of the experiments on turbulent convection [11].

Since the correlation function (9) decreases as r_{12} goes down into the inertial interval, then the flux of θ^4 is constant there and so the other fluxes

$$\langle[(\mathbf{v}_1\cdot\boldsymbol{\nabla}_1)+(\mathbf{v}_2\cdot\boldsymbol{\nabla}_2)]\theta_1^n\theta_2^n\rangle = P_n \ . \tag{10}$$

It means that the probability distribution of temperature (as well as any correlation function) depends, generally speaking, on the infinite set of parameters $(P_2,\ldots,P_n,\ldots)$. The infinite number of restrictions (10) makes the statistics of turbulent convection strongly non-Gaussian. In the experiments [11], it was indeed observed that non-Gaussianity appears at a Rayleigh number that corresponds to the transition from soft to hard turbulence exactly when the Bolgiano-Obukhov scaling appears.

As one can see, all the exponents of the correlation functions (10) are zero and do not coincide with the estimate $\eta-1+2n\zeta$ based on using one scaling exponent of the temperature field. Turbulent convection is thus multiscaling. This explains the absence of a simple scaling in the measurements of probability distribution of temperature difference [12].

The same is true for any tracer (passive or active) in incompressible flow: if the pair correlation function is scale invariant with the exponent ζ_2, then the exponent of two-point correlation function of $2n$th order does not equal $n\zeta_2$. Note that this statement is stronger than that of non-Gaussianity [12–16]: any one-parameter probability distribution (exponent, streched exponent, etc.) is irrelevant. For example, for passive scalar (with $\mathbf{v}$ being independent of θ) Eq. (10) gives, in particular, $\zeta_n=\zeta_2$ for any n [17]. Despite the fact that we have one scaling exponent for any correlation function, the probability distribution function (PDF) should be multiparametric in this case. Indeed, any one-parameter PDF $f(x/\bar{x})$ with $x=\theta_1\theta_2$ and $\bar{x}\propto r_{12}^{\zeta_2}$ gives

$$\langle x^n\rangle = \int x^n f(x/\bar{x})dx \propto \bar{x}^n \propto r_{12}^{n\zeta_2} \ .$$

The steady PDF should depend on the injection rates of all powers of θ that are determined by an external pumping and can be arbitrary.

Coming back to the turbulent convection, one can make the simplest assumption that the spectral density of any integral is mainly determined by its own flux. It gives the following scaling:

$$\langle(\theta_1-\theta_2)^n\rangle \simeq \left(P_n r_{12}^{1/2}\right)^{4n/(4n+1)} \propto r_{12}^{\zeta_n} \ . \tag{11}$$

Let us emphasize that (as distinct from the above results) the formula $\zeta_n=2n/(4n+1)$ is a pure conjecture. One can suggest another possibility that the velocity field is single scaling so that ζ_n =const. It would be interesting to extract the dependence $\zeta(n)$ from the experimental data on turbulent convection.

V. ACTION CASCADE IN CLEBSCH VARIABLES

Kelvin's theorem enables one to rewrite the Euler equation for incompressible fluid as an advection equation [18,19]

$$\frac{\partial a(\mathbf{r},t)}{\partial t} + (\mathbf{v}\cdot\boldsymbol{\nabla})a(\mathbf{r},t) = p(\mathbf{r},t) - d(\mathbf{r},t) \ . \tag{12}$$

Here again p and d are some external pumping and damping, respectively, which may be functionals of a. The fluid velocity is expressed via the complex Clebsch field $a(\mathbf{r},t)$ as follows:

$$\mathbf{v} = i\big(1-\nabla\Delta^{-1}\mathrm{div}\big)(a\nabla a^* - a^*\nabla a) \ . \tag{13}$$

Considering three velocity components with additional restriction of incompressibility is equivalent to considering one complex function $a(\mathbf{r},t)$ for flows with zero helicity [19]. For nonzero helicity (with knotted vortex lines) one should introduce three complex functions [20] which has no influence on what follows. Without p and d, Eq. (12) conserves the integral of any function of $a(\mathbf{r},t)$. We restrict ourselves with real quantities considering the integrals of motion of the kind $I_n=\int|a(\mathbf{r},t)|^{2n}\,d\mathbf{r}$.

Before studying cascades in Clebsch variables one should understand how the boundary conditions in k space (pumping and damping) change while one passes from Clebsch variables to velocity [20]. Since this is a nonlinear transform one may be concerned that a spectrally narrow pumping in one variable might give a pumping distributed over the entire inertial interval in another variable. Whether it is so or not also depends on the behavior of the correlation functions. Fortunately, such a "distributed pumping" is absent for our case. To show this we pass into k space $\mathbf{v}_k = \int \boldsymbol{\psi}_{12} a_1 a_2^* \delta(\mathbf{k} + \mathbf{k}_1 - \mathbf{k}_2) d\mathbf{k}_1 d\mathbf{k}_2$, where $\boldsymbol{\psi}_{12} = \mathbf{k}_1 + \mathbf{k}_2 - (\mathbf{k}_1 - \mathbf{k}_2)(k_1^2 - k_2^2)/|\mathbf{k}_1 - \mathbf{k}_2|^2$. The force $\mathbf{f}_v = d\mathbf{v}/dt$ acting on the velocity field is expressed via p

$$\mathbf{f}_v(\mathbf{k}) = \int \boldsymbol{\psi}_{12}(a_1 p_2^* + p_1 a_2^*)\delta(\mathbf{k} + \mathbf{k}_1 - \mathbf{k}_2) d\mathbf{k}_1 d\mathbf{k}_2 \ .$$

The pumping of energy is determined by the correlation function

$$\langle \mathbf{f}_v(\mathbf{k}) \cdot \mathbf{v}(\mathbf{k}') \rangle = \int (\boldsymbol{\psi}_{12} \cdot \boldsymbol{\psi}_{34}) \langle a_1 a_2^* (a_3 p_4^* + p_3 a_4^*) \rangle \times \delta(\mathbf{k} + \mathbf{k}_1 - \mathbf{k}_2)\delta(\mathbf{k}' + \mathbf{k}_3 - \mathbf{k}_4) \times d\mathbf{k}_1 d\mathbf{k}_2 d\mathbf{k}_3 d\mathbf{k}_4 \ .$$

It is the reducible part $\langle pa^* \rangle \langle aa^* \rangle$ that might be dangerous for the pumping localization in k space. We assume the p pumping to be localized: $\langle p_k a_{k'}^* \rangle = C\delta(\mathbf{k} - \mathbf{k}')\delta(k - \kappa)$. In the right inertial interval $k \gg \kappa$ the reducible part of the correlator gives the following contribution into the energy flux: $\delta P(k) \propto C \int^k q^2 n(q) \psi_{q\kappa}^2 dq$, which is k independent if $n(q)$ drops faster than k^{-3}. We have designated $\langle a_k a_{k'}^* \rangle = n_k \delta(\mathbf{k} - \mathbf{k}')$. Since for the direct (Kolmogorov) cascade $n(q) \propto q^{-13/3}$, then the pumping is negligible and the flux is constant in the inertial interval. One can similarly consider the left inertial interval $k \ll \kappa$ and show that the pumping is absent there. The same consideration can be employed for the external damping (both the viscous small-scale dissipation and a large-scale friction). If the damping is bounded to small (large) scales in Clebsch variables, then it is negligible in the right (left) inertial interval in terms of velocity.

We thus can consider turbulence in Clebsch variables with the scales of pumping and damping being strongly different. To produce I_n, the pumping p should be of non-Hamiltonian nature [20] (similarly, one should have non-potential external force to produce vorticity). One can show that if there is a left inertial interval with a large-scale damping, then the fluxes of I_n are directed up scale opposite the energy flux in three dimensions [21]. Here we consider this hypothetical inverse cascade assuming $L_p \ll r_{12} \ll L_d$. By requiring the flux of the action I_2 to be constant, Zakharov and L'vov [22] postulated that $n(k) \propto k^{-4}$, which corresponds to $\langle |a(r_1) - a(r_2)|^2 \rangle \propto r_{12}$. This corresponds to the energy spectrum $E_k \propto k^{-1}$. Similar to (3), one can consider the fluxes of higher integrals. The only difference is that one should consider d instead of p since the small-scale pumping gives no contribution for larger scales. Since the pair correlation function drops as r_{12} decreases, then "distributed damping" is absent. All the fluxes should be constant in the inertial interval:

$$\langle [(\mathbf{v}_1 \cdot \boldsymbol{\nabla}_1) + (\mathbf{v}_2 \cdot \boldsymbol{\nabla}_2)] a_1^n a_2^{*n} \rangle = P_n \ . \qquad (14)$$

Again, we see that a one-parameter solution with a linear scaling is impossible. The statistics of the field a should be the subject of infinite number of restrictions. This makes irrelevant any consideration of this cascade in the framework of closures or in the one-loop approximation [23]. If the Clebsch correlation functions possess an anomalous scaling this should also be the case for the velocity correlation functions provided such an inverse cascade could exist.

VI. ACKNOWLEDGMENTS

Support of the Rashi Foundation is gratefully acknowledged. I am indebted to V. Lebedev, I. Procaccia, P. Constantin, E. Siggia, and R. Kraichnan for useful remarks.

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